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Group-theoretical approach to a non-central extension of the Kepler–Coulomb problem

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Abstract
Bound and scattering states of a non-central extension of the three-dimensional Kepler–Coulomb Hamiltonian are worked out analytically within the framework of the potential groups of the problem, $SO(7)$ for bound states and $SO(6, 1)$ for scattering states. In the latter case, the S-matrix is calculated by the method of intertwining operators.

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1. Introduction

In classical mechanics, the reduced Kepler problem has been known for more than two centuries [1] to admit seven integrals of motion. These are the total angular momentum, the Laplace–Runge–Lenz (or Hermann–Bernoulli–Laplace) vector and the total energy. Since there are two relationships between them (see, for example, [2]) only five of the integrals of motion are independent. In general, a closed system with N degrees of freedom can have at most $2N - 1$ independent integrals of motion [3]. According to the Liouville theorem, the system is completely integrable if it allows N integrals of motion (including the Hamiltonian) that are independent and in involution (i.e. Poisson brackets of any two integrals are zero). The system is called superintegrable if there exist q, $1 \leqslant q \leqslant N - 1$, additional independent integrals of motion. The cases $q = 1$ and $q = N - 1$ correspond to minimal and maximal superintegrability, respectively. In quantum mechanics, the definitions of complete integrability and superintegrability are same, but Poisson brackets are replaced by commutators.

The first systematic search for quantum integrable one-particle systems with scalar potentials was begun by Smorodinsky and co-workers in [4–6] and continued by Evans in [7]. It was restricted to the cases when integrals of motion are first- or second-order polynomials in the momenta. They found all superintegrable systems in two and three dimensions with at most second-order integrals of motion. It turns out that they possess properties making them

of special interest: for instance, all these potentials admit the separation of variables in several coordinate systems and possess dynamical symmetries responsible for the separability of the Schrödinger equation. The history of this problem and some results may be found in [8].

It is well known that the first quantum study of the hydrogen atom [9] was based upon the algebra generated by integrals of motion, before the Schrödinger equation was published. Later, Fock [10] and Bargmann [11] recognized that the angular momentum and the Laplace–Runge–Lenz vector generate the Lie algebra of $SO(4)$ in the subspace of negative energies and the Lie algebra of $SO(3,1)$ in the subspace of positive energies. It was realized that the 'accidental' degeneracies, i.e. degeneracies not connected with geometrical $SO(3)$ symmetries of the Hamiltonian, are due to the invariance group $SO(4)$. Moreover, the separation of variables in parabolic coordinates was related to the Laplace–Runge–Lenz vector [11]. Zwanziger [12] later showed that the algebra of $SO(3,1)$ may be used to calculate the Coulomb phase-shifts. Ever since, invariance algebras have been determined for many quantum mechanical systems. The best known of these systems are the oscillator [13] and the MICZ-Kepler system [14, 15]. This is a situation in which the Hamiltonian H of the system belongs to the centre of the enveloping algebra of some group G, i.e.

$$H = f(C), \tag{1}$$

where C is the Casimir operator of the invariance group G. For example, in the Coulomb bound-state problem, $H = -\gamma^2/2(C+1)$, where C is a Casimir operator of $SO(4)$.

But it could happen that the Hamiltonian H_ν can be related to the Casimir operator C as

$$H_\nu = f(C)|_{\mathfrak{H}_\nu}, \tag{2}$$

where $\mathfrak{H}_\nu$ is subspace occurring in the subgroup reduction and $|_{\mathfrak{H}_\nu}$ denotes the restriction to $\mathfrak{H}_\nu$. In this case the group G describes the same energy states of a family of Hamiltonians H_ν with different potential strength. (This is why the present group G designated the potential group [16].) Such an approach was proposed by Ghirardi [17], who worked it out in detail for the Scarf potential [18]. It is similar to the approach of Olshanetsky and Perelomov [19, 20], where quantum integrable systems are related to a radial part of the Laplace operator on homogeneous spaces (i.e. to a radial part of the Casimir operator of second order) of Lie groups.

Reference [21] proposed a method that permits purely algebraic calculations of S-matrices for the systems whose Hamiltonians are related to the Casimir operators C of some Lie group G as (1) or (2). Namely, the S-matrices for the systems under consideration are associated with intertwining operators A between Weyl equivalent representations U^χ and $U^{\tilde{\chi}}$ of G as

$$S = A \tag{3}$$

or

$$S = A|_{\mathfrak{H}_\nu}, \tag{4}$$

respectively. (The representations U and $U^{\tilde{\chi}}$ have the same Casimir eigenvalues. Such representations are called Weyl equivalent.) At this stage we note that the operator A is said to intertwine the representations U^χ and $U^{\tilde{\chi}}$ of the group G if the relation

$$AU^\chi(g) = U^{\tilde{\chi}}(g)A \qquad \text{for all} \quad g \in G \tag{5}$$

or

$$A\mathrm{d}U^\chi(b) = \mathrm{d}U^{\tilde{\chi}}(b)A \qquad \text{for all} \quad b \in \mathfrak{g} \tag{6}$$

holds, where $\mathrm{d}U^\chi$ and $\mathrm{d}U^{\tilde{\chi}}$ are the corresponding representations of the algebra $\mathfrak{g}$ of G. Equations (5) and (6) have high restriction power, determining the intertwining operator up to a constant.

The potential group approach has been proven to be useful in a variety of problems in one dimension. Recently, it has been used to describe some potentials [22–25] classified in [6]. In [23] it has been shown that the superposition of the Coulomb potential with one barrier term [6] could be related to the potential group $SO(5)$. Scattering amplitudes for such a system are worked out in detail in [24] by using an intertwining operator [21] between two Weyl-equivalent unitary irreducible representations of the $SO(5, 1)$ potential group.

The subject of the present work will be the simultaneous description of bound and scattering states of a quantum mechanical system with the Hamiltonian

$$H = -\frac{1}{2}\nabla^2 - \frac{\gamma}{r} + \frac{s_1^2 - 1/4}{2x^2} + \frac{s_2^2 - 1/4}{2y^2} + \frac{s_3^2 - 1/4}{2z^2} \tag{7}$$

written in units $\hbar = m = 1$, where $s_i = 0, 1, 2, \ldots$. We show that

$$H = -\frac{\gamma^2}{2\left(C + \frac{25}{4}\right)\big|_{\mathcal{H}_{s_1 s_2 s_3}}}, \tag{8}$$

where C is a Casimir operator of $SO(7)$ (for bound states) or $SO(6, 1)$ (for scattering states).

This system was proved to be minimally superintegrable [6], since four integrals of motions were explicitly derived, as a consequence of the separability of the related Schrödinger equation in two coordinate systems. But in [26], it has been shown that the classical counterpart of Hamiltonian (7) is maximally superintegrable, i.e. it admits five independent integrals of motion, including the Hamiltonian: four of them derive from separability of the related Hamilton–Jacobi equation in different coordinate systems, but the fifth integral, first discussed in [26], is not connected with separability. Moreover, this last integral is quartic in the momenta, while the other three are quadratic, and has been rederived in [27] as an example of an application of a more general technique.

2. General formalism

Let us start the discussion with the fact that the generators of UIR of $SO(7)$ (or $SO(6, 1)$) are 21 independent Hermitian operators $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu, \nu = 1, 2, \ldots, 7)$ which obey the commutation relations

$$[M_{\mu\nu}, M_{\sigma\lambda}] = \mathrm{i}(g_{\mu\sigma} M_{\nu\lambda} + g_{\nu\lambda} M_{\mu\sigma} - g_{\mu\lambda} M_{\nu\sigma} - g_{\nu\sigma} M_{\mu\lambda}), \tag{9}$$

where

$$\begin{aligned} g_{\mu\nu} &= (+, +, \ldots, +, +) \qquad \text{for} \quad SO(7) \\ g_{\mu\nu} &= (+, +, \ldots, +, -) \qquad \text{for} \quad SO(6, 1). \end{aligned} \tag{10}$$

There are three independent Casimir invariants which are identically multiple of the unit in each UIR. In the case of most degenerate representations, they are identically zero, with the exception of the second-order Casimir operator

$$C = \frac{1}{2} \sum_{\mu,\nu=1}^{7} M_\mu^\nu M_\nu^\mu, \tag{11}$$

It is well known that the most degenerate representation of the algebra $so(7)$ $(so(6, 1))$ can be realized in the Hilbert space spanned by negative-energy (positive-energy)

states corresponding to a fixed eigenvalue of the Coulomb Hamiltonian H^{Coul} in six dimensions:

$$H^{\text{Coul}} = \frac{1}{2}p^2 - \frac{\gamma}{\sqrt{x^2}}, \qquad \gamma > 0, \tag{12}$$

where $x = (x_1, x_2, \ldots, x_6) \in R^6$, $p_j = -\mathrm{i}\frac{\partial}{\partial x_j}$ $(j = 1, \ldots, 6)$, $x^2 = \sum_{i=1}^6 x_i x_i$, $p^2 = \sum_{i=1}^6 p_i p_i$. (We are using units with $M = \hbar = 1$.) However, in order to be able to write relation (2) we introduce the following realization;

$$M_{ij} = \lambda(x) \circ (x_i p_j - x_j p_i) \circ \lambda^{-1}(x), \tag{13}$$

$$M_{i7} = -M_{7i} = |2h|^{-\frac{1}{2}}\lambda(x) \circ \left[x_i p^2 - p_i(x \cdot p) + \mathrm{i}\frac{5}{2}p_i - \frac{\gamma x_i}{x^2}\right] \circ \lambda^{-1}(x), \quad (i, j = 1, \ldots, 6), \tag{14}$$

where

$$\lambda(x) = \left[\left(x_1^2 + x_2^2\right)\left(x_3^2 + x_4^2\right)\left(x_5^2 + x_6^2\right)\right]^{1/4} \tag{15}$$

and

$$h = \lambda(x) \circ \left(\frac{1}{2}p^2 - \frac{\gamma}{\sqrt{x^2}}\right) \circ \lambda^{-1}(x). \tag{16}$$

The generators (13)–(14) act in the eigenspace of h equipped with the scalar product

$$(\phi_1, \phi_2) = \int_{R^6} \phi_1^*(x)\phi_2(x)\,\mathrm{d}\mu(x), \qquad x \in R^6, \tag{17}$$

where $\mathrm{d}\mu(x) = \lambda^{-2}(x)\,\mathrm{d}x_1\,\mathrm{d}x_2 \cdots \mathrm{d}x_6$.

This representation, of course, is unitarily equivalent to the representation constructed in the eigenspace of the Coulomb Hamiltonian H^{Coul} in six dimensions. The unitary mapping W which realizes the equivalence is given by

$$W: \quad \Psi^{\text{Coul}} \to \Phi = \lambda(x)\Psi^{\text{Coul}}. \tag{18}$$

The operators (13) and (14) provide the most degenerate representations of $SO(7)$ if h is negative definite and of $SO(6, 1)$ if h is positive definite. More precisely, they define the most degenerate (symmetric) UIR of $SO(7)$ specified by the integer number $j = 0, 1, \ldots$ (when h is negative definite) and the most degenerate principal series representations of $SO(6, 1)$ labelled by the complex number $j = -\frac{5}{2} + \mathrm{i}\rho$, $\rho > 0$ (when h is positive definite). If we compute the second-order Casimir operator (11), it becomes

$$C = -\frac{25}{4} - \frac{\gamma^2}{2h}. \tag{19}$$

Let us consider the reduction corresponding to the group chain $G \supset SO(6) \supset SO(4) \times SO(2) \supset SO(2) \times SO(2) \times SO(2)$, where G is $SO(6, 1)$ or $SO(7)$. Then, the basis functions can be characterized by the Casimir operators of the chain of groups

$$\begin{aligned}
&C|j; lM\rangle = j\,(j+5)\,|j; lM\rangle \\
&C^{SO(6)}|j; lM\rangle = l\,(l+4)\,|j; lM\rangle \\
&C^{SO(4)}|j; lM\rangle = m\,(m+2)\,|j; lM\rangle \\
&C^{SO(2)_1}|j; lM\rangle = s_1^2|j; lM\rangle \\
&C^{SO(2)_2}|j; lM\rangle = s_2^2|j; lM\rangle \\
&C^{SO(2)_3}|j; lM\rangle = s_3^2|j; lM\rangle,
\end{aligned} \tag{20}$$

where M is a collective index (m, s_1, s_2, s_3) and

$$C^{SO(6)} = \frac{1}{2}\sum_{i,j=1}^{6} M_{ij}^2, \qquad C^{SO(4)} = \frac{1}{2}\sum_{i,j=1}^{4} M_{ij}^2,$$
$$C^{SO(2)_1} = M_{12}^2, \qquad C^{SO(2)_2} = M_{34}^2, \qquad C^{SO(2)_3} = M_{56}^2. \tag{21}$$

According to this, we introduce in place of $x_1, x_2, \ldots, x_6$ the variables $r, \theta, \varphi, \alpha_1, \alpha_2, \alpha_3$ via $x_i = rn_i$ with

$$n_1 = \sin\theta \sin\varphi \sin\alpha_1, \qquad n_2 = \sin\theta \sin\varphi \cos\alpha_1$$
$$n_3 = \sin\theta \cos\varphi \sin\alpha_2, \qquad n_4 = \sin\theta \cos\varphi \cos\alpha_2 \tag{22}$$
$$n_5 = \cos\theta \sin\alpha_3, \qquad n_6 = \cos\theta \cos\alpha_3,$$

where $0 \leqslant r < \infty, 0 \leqslant \theta, \varphi \leqslant \frac{\pi}{2}$ and $0 \leqslant \alpha_1, \alpha_2, \alpha_3 \leqslant 2\pi$. If we compute the operator $\gamma^2/\left(C + \frac{25}{4}\right)$ for this parametrization, it becomes

$$\frac{\gamma^2}{C+\frac{25}{4}} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) + \frac{1}{r^2\sin^2\theta\sin^2\varphi}$$
$$\times\left(\frac{1}{4} + \frac{\partial^2}{\partial\alpha_1^2}\right) + \frac{1}{r^2\sin^2\theta\cos^2\varphi}\left(\frac{1}{4} + \frac{\partial^2}{\partial\alpha_2^2}\right) + \frac{1}{r^2\cos^2\theta}\left(\frac{1}{4} + \frac{\partial^2}{\partial\alpha_3^2}\right). \tag{23}$$

Let $\mathcal{H}_{s_1s_2s_3}$ be a subspace spanned by $|j; lM\rangle$ with fixed s_1, s_2 and s_3. Thus, the operator (23) restricted to this subspace becomes a differential operator in r, θ and φ; it turns out that

$$\left.\frac{\gamma^2}{C+\frac{25}{4}}\right|_{\mathcal{H}_{s_1s_2s_3}} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$
$$+ \frac{1/4 - s_1^2}{r^2\sin^2\theta\sin^2\varphi} + \frac{1/4 - s_2^2}{r^2\sin^2\theta\cos^2\varphi} + \frac{1/4 - s_3^2}{r^2\cos^2\theta} \tag{24}$$

with $s_i = 0, \pm1, \pm2\ldots$, where we have used that

$$C^{SO(2)_i} = -\frac{\partial^2}{\partial\alpha_i^2}, \qquad i = 1, 2, 3.$$

Hence, the Hamiltonian

$$H = -\frac{1}{2}\nabla^2 - \frac{\gamma}{r} + \frac{s_1^2 - 1/4}{2r^2\sin^2\theta\sin^2\varphi} + \frac{s_2^2 - 1/4}{2r^2\sin^2\theta\cos^2\varphi} + \frac{s_3^2 - 1/4}{2r^2\cos^2\theta} \tag{25}$$

can be described in terms of the potential groups $SO(7)$ and $SO(6, 1)$ since

$$H = -\left.\frac{\gamma^2}{C+\frac{25}{4}}\right|_{\mathcal{H}_{s_1s_2s_3}},$$

as mentioned in section 1 (formula (8)). (Due to the symmetry $s_i \to -s_i$ in the Hamiltonian (25), without loss of generality, we may assume that s_1, s_2 and s_3 are non-negative integers.) Note that the $SO(2)$ subgroups are related to potential strength.

At this point, it is worth pointing out that Hamiltonian (25) does not contain the pure Coulomb potential as a particular case, within the framework of the $SO(7)$ and $SO(6, 1)$ symmetries considered in the present work. In order to restore it, it is necessary to resort to larger symmetry groups, for example, $SO\,(10)$ and $SO\,(9, 1)$ and use the decomposition chain $G \supset SO\,(9) \supset SO(6) \times SO\,(3) \supset SO\,(3) \times SO\,(3) \times SO\,(3)$, where now the $SO(3)$ subgroups are related to potential strength.

Here again, polar coordinates are used:

$$x = (\sin\theta \sin\varphi e_1, \sin\theta \cos\varphi e_2, \cos\theta e_3),$$

where $x \in R^9$, $e_i = (\sin\alpha_i \sin\beta_i, \sin\alpha_i \cos\beta_i, \cos\alpha_i)$, $i = 1, 2, 3$. Then, a procedure similar to that described above would lead to the Hamiltonian

$$H = -\frac{1}{2}\nabla^2 - \frac{\gamma}{r} + \frac{l_1(l_1+1)}{2r^2 \sin^2\theta \sin^2\varphi} + \frac{l_2(l_2+1)}{2r^2 \sin^2\theta \cos^2\varphi} + \frac{l_3(l_3+1)}{2r^2 \cos^2\theta}$$

where l_i ($i = 1, 2, 3$) are integers and are allowed to take the null value, thus restoring the pure Coulomb potential.

Finally, we note that the operators

$$\begin{aligned} I_1 &= \mathbf{L}^2 + \frac{s_1^2 - \frac{1}{4}}{\sin^2\theta \sin^2\varphi} + \frac{s_2^2 - \frac{1}{4}}{\sin^2\theta \cos^2\varphi} + \frac{s_3^2 - \frac{1}{4}}{\cos^2\theta} \\ I_2 &= L_z^2 + \frac{s_1^2 - \frac{1}{4}}{\sin^2\varphi} + \frac{s_2^2 - \frac{1}{4}}{\cos^2\varphi}, \end{aligned} \tag{26}$$

where $\mathbf{L}^2$ and L_z^2 are the squares of angular momentum and of its projection on the third axis, commute with the Hamiltonian. These integrals of motion are related to the Casimir operators of $SO(6)$ and its $SO(4)$ subgroup in the sense that

$$I_1 = C^{SO(6)}|_{\mathcal{H}_{s_1 s_2 s_3}}, \qquad I_2 = C^{SO(4)}|_{\mathcal{H}_{s_1 s_2 s_3}}, \tag{27}$$

where

$$\begin{aligned} C^{SO(6)} = &-\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right) \\ &- \frac{1}{\sin^2\theta \sin^2\varphi}\left(\frac{1}{4} + \frac{\partial^2}{\partial\alpha_1^2}\right) - \frac{1}{\sin^2\theta \cos^2\varphi}\left(\frac{1}{4} + \frac{\partial^2}{\partial\alpha_2^2}\right) - \frac{1}{\cos^2\theta}\left(\frac{1}{4} + \frac{\partial^2}{\partial\alpha_3^2}\right) \end{aligned}$$

and

$$C^{SO(4)} = -\frac{\partial^2}{\partial\varphi^2} - \frac{1}{\sin^2\varphi}\left(\frac{1}{4} + \frac{\partial^2}{\partial\alpha_1^2}\right) - \frac{1}{\cos^2\varphi}\left(\frac{1}{4} + \frac{\partial^2}{\partial\alpha_2^2}\right).$$

2.1. Bound states

The bound-state spectrum can now be easily obtained if we note that the eigenvalue of the Casimir operator C of the potential group $SO(7)$ is $j(j+5)$. We then find

$$E = -\frac{\gamma^2}{2\left(j + \frac{5}{2}\right)^2}, \tag{28}$$

where $j = s_1 + s_2 + s_3 + 2k_1 + 2k_2 + n(k_1, k_2, n = 0, 1, 2, \ldots)$. It is easy to check that states (28) have degeneracy $\frac{d(d+1)}{2}$, where $d = \left[\frac{j - s_1 - s_2 - s_3}{2}\right] + 1$, and $[q]$ is the largest integer less than or equal to q.

We give for reference the expression of the bound-state wavefunctions

$$\psi(x) = \mathcal{R}_{jl}(r)\, \mathcal{Y}_{lM}(\theta, \varphi), \tag{29}$$

where $\mathcal{R}_{jl}(r)$ is the radial part of the wavefunction, while $\mathcal{Y}_{lM}(\theta, \varphi)$ is the angular part of it:

$$\mathcal{R}_{jl}(r) = c u^{l+\frac{3}{2}} \mathrm{e}^{-\frac{u}{2}} L_n^{2l+4}(u), \qquad u = 2\gamma r \Big/ \left(j + \tfrac{5}{2}\right) \tag{30}$$

with $n = j - l$ ($n = 0, 1, 2, \ldots$),

$$c = \left(\frac{2\gamma}{j + \frac{5}{2}}\right)^3 \left[\frac{\Gamma(j - l + 1)}{2\left(j + \frac{5}{2}\right)\Gamma(j + l + 5)}\right]^{\frac{1}{2}} \tag{31}$$

and

$$\mathcal{Y}_{lM}(\theta,\varphi)=\chi\,\sin^{m+1}\theta\cos^{s_3+\frac{1}{2}}\theta\sin^{s_1+\frac{1}{2}}\varphi\cos^{s_2+\frac{1}{2}}\varphi P_{k_1}^{(m+1,s_3)}(\cos 2\theta)\,P_{k_2}^{(s_1,s_2)}(\cos 2\varphi) \tag{32}$$

with $2k_1=l-m-s_3$, $2k_2=m-s_1-s_2$ $(k_1,k_2=0,1,2,\ldots)$ and

$$\chi=\left[\frac{\Gamma\left(\frac{1}{2}(l+m+s_3+4)\right)\Gamma\left(\frac{1}{2}(l-m-s_3+2)\right)\Gamma\left(\frac{1}{2}(m+s_1+s_2+2)\right)}{\Gamma\left(\frac{1}{2}(l+m-s_3+4)\right)\Gamma\left(\frac{1}{2}(l-m+s_3+2)\right)\Gamma\left(\frac{1}{2}(m+s_1-s_2+2)\right)}\right]^{\frac{1}{2}}$$
$$\times\left[\frac{\Gamma\left(\frac{1}{2}(m-s_1-s_2+2)\right)}{\Gamma\left(\frac{1}{2}(m-s_1+s_2+2)\right)}(2l+4)(2m+2)\right]^{\frac{1}{2}}. \tag{33}$$

Here, L_n^{α} and $P_n^{(\alpha,\beta)}$ are the Laguerre and Jacobi polynomials, respectively. It is worth noting that (see the appendix) the $\mathcal{Y}_{lM}$ functions are related to five-dimensional spherical harmonics $Y_{lM}(n)$ (see section 10.5 of [28]) in polyspherical coordinates, while $\mathcal{R}_{jl}$ is related to the radial part of the six-dimensional Coulomb wavefunction [29] as

$$\mathcal{R}_{jl}(r)=r^{\frac{3}{2}}\mathcal{R}_{jl}^{\mathrm{Coul}}(r).$$

2.2. *Scattering states*

Once the group structure of the problem has been recognized, the associated S-matrix can be computed by using equations (3)–(6). This requires knowledge of matrices $\langle l'M'|A|lM\rangle$ that intertwine Weyl-equivalent representations of $SO(6,1)$ in the bases corresponding to the $SO(6,1)\supset SO(6)\supset SO(4)\times SO(2)\supset SO(2)\times SO(2)\times SO(2)$ reduction. One has (see the appendix)

$$\langle l'M'|A|lM\rangle=A_l\delta_{ll'}\delta_{MM'}, \tag{34}$$

where

$$A_l=\frac{\Gamma\left(\frac{5}{2}+\mathrm{i}\rho+l\right)}{\Gamma\left(\frac{5}{2}-\mathrm{i}\rho+l\right)}. \tag{35}$$

According to this, we have

$$S(\theta,\varphi;\theta',\varphi')=\sum_{lM}A_l\mathcal{Y}_{lM}(\theta,\varphi)\mathcal{Y}_{lM}^*(\theta',\varphi'). \tag{36}$$

Thus, the scattering amplitude, $f(\theta,\varphi;\theta',\varphi')$, is defined by

$$f(\theta,\varphi;\theta',\varphi')=\frac{2\pi}{ip}\sum_{lM}(A_l-1)\,\mathcal{Y}_{lM}(\theta,\varphi)\,\mathcal{Y}_{lM}^*(\theta',\varphi'). \tag{37}$$

Since

$$\sum_{lM}\mathcal{Y}_{lM}(\theta,\varphi)\,\mathcal{Y}_{lM}^*(\theta',\varphi')=\delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi')$$

we can omit unity in the brackets of formula (37) when $\theta\neq\theta'$, $\varphi\neq\varphi'$, leaving

$$f(\theta,\varphi;\theta',\varphi')=\frac{2\pi}{\mathrm{i}p}\sum_{lM}A_l\mathcal{Y}_{lM}(\theta,\varphi)\,\mathcal{Y}_{lM}^*(\theta',\varphi'). \tag{38}$$

Moreover, formulas (A.8), (A.11) and (A.7) imply the following integral representation of the scattering amplitude:

$$f(\theta,\varphi;\theta',\varphi')=\frac{2\pi}{\mathrm{i}p}\eta\sqrt{b}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}(1-a\sin\theta\sin\theta'-\cos\theta\cos\theta'\cos\alpha_3)^{-\frac{5}{2}-\mathrm{i}\rho}$$
$$\times\exp(-\mathrm{i}s_1\alpha_1-\mathrm{i}s_2\alpha_2-\mathrm{i}s_3\alpha_3)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3, \tag{39}$$

where

$$a = \sin\varphi \sin\varphi' \cos\alpha_1 + \cos\varphi \cos\varphi' \cos\alpha_2 \tag{40}$$

and

$$b = \sin\theta \sin\theta' \sin\varphi \sin\varphi' \tag{41}$$

and η is defined in formula (A.5) of the appendix.

3. Conclusions and outlook

We have shown in the present work, based on the potential group approach, how a non-central extension of the Coulomb Hamiltonian, considered in the literature as an example of maximal superintegrability, can be worked out in a fully analytic way, with bound states described by most degenerate representations of $SO(7)$ and scattering states by most degenerate representations of $SO(6, 1)$. The subfamily of the generalized Coulomb problem described does not include the pure Coulomb potential; in order to restore it, the symmetries could be enlarged to $SO(10)$ and $SO(9, 1)$, respectively.

The generation of solvable non-central potentials via the potential group approach is quite general and not limited to the orthogonal and pseudo-orthogonal groups of interest to the Coulomb problem. An example of a non-central extension of the harmonic oscillator with U (4) symmetry has been discussed in [22], while a non-central extension of the null potential with E (4) symmetry has been worked out in [25]. Other cases of physical interest with more complicated symmetries will be considered for future work.

Appendix. Calculation of the matrix elements of A

Here we calculate the matrix elements of A which intertwine Weyl-equivalent representations of $SO(6, 1)$ or $\mathfrak{so}(6, 1)$ in the bases corresponding to $SO(6, 1) \supset SO(6) \supset SO(4) \times SO(2) \supset SO(2) \times SO(2) \times SO(2)$ reduction. We find it expedient to use, for this purpose, equation (6).

We shall start with the fact that the most degenerate principal series representations of $SO(6, 1)$ can be realized on $\mathcal{L}_2(S^5)$ (see section 9.2.1 of [28]):

$$U_j(g) f(n) = (\omega_g)^j f(n_g), \qquad n \in S^5, \tag{A.1}$$

where

$$\omega_g = \sum_{i=1}^{6} g_{7i}^{-1} n_i + g_{77}, \qquad (n_g)_k = \frac{\sum_{i=1}^{6} g_{ki}^{-1} n_i + g_{k7}}{\sum_{i=1}^{6} g_{7i}^{-1} n_i + g_{77}}.$$

The operator A defined by

$$(Af)(n) = \int K(n, n') f(n')\, \mathrm{d}n' \tag{A.2}$$

intertwines representations j and $-5 - j$, if

$$K(n_g, n'_g) = (\omega_g)^{5+j} (\omega'_g)^{5+j} K(n, n'). \tag{A.3}$$

The kernel, K, is uniquely determined by equation (A.3) up to a constant and is given by

$$K(n, n') = \eta(1 - n \cdot n')^{-5-j}. \tag{A.4}$$

with

$$\eta = 2^{-\frac{5}{2}+\mathrm{i}\rho}\frac{\Gamma\left(\frac{5}{2}+\mathrm{i}\rho\right)}{\pi^{\frac{5}{2}}\Gamma(-\mathrm{i}\rho)} \tag{A.5}$$

With this factor the operator A becomes unitary for $j = -\frac{5}{2} + \mathrm{i}\rho$ (see equation (A.9)).

Taking into account the fact that five-dimensional spherical harmonics Y_{lM} of degree l [28] forms a basis in $\mathcal{L}_2(S^5)$, corresponding to the above reduction, we have the following integral representation for the matrix elements of A:

$$\langle l'M'|A|lM\rangle = \int \mathcal{K}(n,n')Y^*_{l'M'}(n')Y_{lM}(n)\,\mathrm{d}n\,\mathrm{d}n', \tag{A.6}$$

where $\mathrm{d}n = \sin^3\theta\cos\theta\sin\varphi\cos\varphi\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}\alpha_3\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_1$ for n as in (22) and

$$Y_{lM}(n) = \mathcal{Y}_{lM}(\theta,\varphi)\prod_{j=1}^{3}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}s_j\alpha_j}. \tag{A.7}$$

By using the expansion

$$\eta(1-n\cdot n')^{-\frac{5}{2}-\mathrm{i}\rho} = \frac{1}{2\pi^3}\sum_{v=0}^{\infty}(v+2)\frac{\Gamma\left(\frac{5}{2}+\mathrm{i}\rho+v\right)}{\Gamma\left(\frac{5}{2}-\mathrm{i}\rho+v\right)}C^2_v(n\cdot n'), \tag{A.8}$$

we have

$$\langle l'M'|A|lM\rangle = A_l\delta_{ll'}\delta_{MM'} \tag{A.9}$$

with

$$A_l = \frac{\Gamma\left(\frac{5}{2}+\mathrm{i}\rho+l\right)}{\Gamma\left(\frac{5}{2}-\mathrm{i}\rho+l\right)}. \tag{A.10}$$

In arriving at equation (A.9) we have used the addition formula

$$C^2_v(n\cdot n') = \frac{2\pi^3}{v+2}\sum_M Y_{vM}(n)Y^*_{vM}(n'). \tag{A.11}$$

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